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MODELING OF STRATIFIED TWO-PHASE FLOW IN PIPES, PUMPS AND OTHER DEVICES

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Abstract—A model for stratified two-phase flow in pipes, pumps and other devices is presented. Using the assumption of a hydrostatic distribution of pressure over the cross section of a pipe, the effects of stratification are taken into account by means of specific terms in momentum and energy equations. These terms represent the pressure interactions between phases. Four concrete examples (classical earth gravity stratified flow, stratified flow in a regularly curved pipe, annular stratified flow caused by pre- or postrotation, and stratified flow in the impeller of a pump) and a comparison with an experiment are presented.

1. INTRODUCTION

The flow of a two-phase mixture in a pump out of nominal delivery is essentially multidimensional. As a result, classical two-phase flow pump models (cf. Mikielewiez *et al.* 1978) do not deal with what occurs inside pumps and use only overall balance equations and head degradation correlations. This method, which gives fairly good results, needs testing with mock-ups of each type of pump. It yields a prediction of the performances, but not an explanation of why these performances are obtained. The important effects of void fraction, relative volumetric flow rate and specific speed (Kastner *et al.* 1983) are not explained. Consequently, the reliability of such models in full size conditions is not clearly assured.

A conclusion of some experimenters (e.g. Rundstadler & Patel 1978) is that the head degradation of pumps in two-phase flow seems to be caused by the stratification in the impeller. This stratification may be caused by Coriolis forces, centrifugal forces and/or curvature effects (earth's gravity is usually negligible with respect to these effects).

If the flow is stratified, and especially if the stratified flow is torrential (see section VI), the energy of the pump motor is transferred to the liquid phase as kinetic energy. Consequently, the liquid phase accelerates and the vapor phase slows down. In the maladapted diffuser, the conversion into pressure of the high kinetic energy of the liquid phase in the presence of slow vapor is not efficient. Therefore, no significant pressure rise occurs in the impeller and diffuser, and the head is greatly degraded.

On the other hand, if the flow is not stratified, interfacial friction and added mass effects are important and the slip ratio remains probably fairly close to 1. The energy of the pump motor is transferred to the flow as pressure head, and the head is only slightly degraded.

For this reason, a model of two-phase flow inside pumps, using an axial description of the flow and including stratification effects, has been done.

Grison & Lauro (1978) developed such an axial model especially for critical flow in pumps. They obtained fairly good results in even far from nominal conditions, but did not pay special attention to stratification.

It is hoped that an axial model, which includes as many multidimensional effects (e.g. stratification) as possible, will enable the main phenomena to be understood and predicted. Of course, this model should be used only when the flow is more or less one-dimensional, that is to say only in the first and third quadrants (normal flow and normal speed of rotation or inverse flow and inverse speed of rotation).

In this paper, stratification in the most general case will be studied. After explaining the basic hypothesis and the main assumptions, we will establish momentum and energy equations. Four examples that are useful for pump study will be presented: classic stratified flow, stratification in a curved pipe, stratification caused by pre- or postrotation in the suction duct of a pump and stratification in the impeller of a pump. Comparison with a very simple experiment will also be presented.

2. BASIC PRINCIPLES

The component normal to the tube axis of the forces per unit of mass acting on the flow is named the apparent transverse gravity. The main hypothesis used in stratified flow, is that a hydrostatic distribution of pressure is present in any cross section of the duct created by the apparent transverse gravity (Hypothesis H1). The main advantages and disadvantages of this model result from this hypothesis.

Four other assumptions are made:

H2. The effects of variations of the phase density ρ_v and ρ_l over the cross section are neglected.

H3. Continuity of pressure across the interface (no effects of surface tension or pressure jump induced by mass transfer) is assumed.

H4. Each phase is continuous in any cross section (for each phase, any two points in the phase can be joined by a continuous path lying entirely in the phase).

H5. The interface is isobaric (using H1, H2, H3, H4, one can show that this assumption is equivalent to assuming no local slip at the interface).

With these assumptions, the Euler equations reduce to the forms

$$\frac{\partial p_k}{\partial x} = \rho_k F_{k,x}(\alpha, V_v, V_l, M, x, y, z, t, \ldots),$$
$$\frac{\partial p_k}{\partial y} = \rho_k F_{k,y}(\alpha, V_v, V_l, M, x, y, z, t, \ldots), \qquad [2.1]$$

where k is the phase index ($k=v$ for vapor and $k=l$ for liquid), ρ_k is the density, p_k is the local phase pressure, $F_{k,x}$ and $F_{k,y}$ are the apparent transverse gravity projected over x and y axes dependent on the void fraction α, V_v and V_l are the instantaneous space averaged vapor and liquid velocities, M is the total angular momentum, x and y are the transverse coordinates, z is the axial coordinate and t is the time.

The apparent transverse gravity may be caused, for example, by

— earth's gravity,
— curvature of the pipe,
— pre- or postrotation of a pump,
— Coriolis and centrifugal forces.

Let P_i be the pressure at a point i of the interface, with coordinates (X_i, Y_i). (P_i is well defined, as a consequence of assumptions H3 and H5).

From [2.1], H3 and H4, we have

$$p_k(X,Y) = P_i(X_i, Y_i) + \int_{X_i}^{X} \frac{\partial p_k}{\partial x}\,dx + \int_{Y_i}^{Y} \frac{\partial p_k}{\partial y}\,dy, \qquad [2.2]$$

where $p_k(X, Y)$ is the local pressure at the point (X, Y).

Let δ_k be defined by

$$\delta_k \triangleq \int_{X_i}^{X} \frac{\partial p_k}{\partial x}\,dx + \int_{Y_i}^{Y} \frac{\partial p_k}{\partial y}\,dy. \qquad [2.3]$$

Using the notation $< X_k >$ for instantaneous phase averaged value in a cross section of any variable X_k (Delhaye 1981), the phase averaged value of δ_k is denoted by Δ_k

$$\Delta_k \triangleq <\delta_k> . \qquad [2.4]$$

As a consequence of assumption H5, Δ_k is independent of the choice of point i. Therefore, the mean pressure in phase k, $<p_k>$, is

$$<p_k> = P_i + \Delta_k \quad . \tag{2.5}$$

For example, in the particular case where $F_{k,x}$ and $F_{k,y}$ are independent of x and y, we have

$$\Delta_k = \rho_k \left[F_{k,x}(x_{G,k} - X_i) + F_{k,y}(y_{G,k} - Y_i)\right], \tag{2.6}$$

where $(x_{G,k}, y_{G,k})$ are the coordinates of the center of inertia of the phase k (cf. figure 1).

In particular, one can show that there are two equilibrium configurations for the flow, one of which is stable (minimum of potential energy) and determined by the condition

$$\Delta_v < 0 \text{ and } \Delta_l > 0 \quad . \tag{2.7}$$

Since less dense phase is at the top, its pressure is less than that at the interface. Likewise, the heavier phase at the bottom has a pressure greater than that at the interface.

Defining P as the mean pressure in the flow:

$$P \overset{\Delta}{=} \alpha <p_v> + (1-\alpha) <p_l> \,, \tag{2.8}$$

it can be shown that

$$P = P_i + \alpha \, \Delta_v + (1-\alpha) \, \Delta_l \quad . \tag{2.9}$$

We have

$$<P_k> = P + (1-\alpha_k) \, \epsilon_k(\Delta_v - \Delta_l) \tag{2.10}$$

with

$$\epsilon_v = 1 \text{ and } \epsilon_l = -1 \quad , \tag{2.11}$$

$$\alpha_v = \alpha \text{ and } \alpha_l = 1 - \alpha \quad . \tag{2.12}$$

3. MOMENTUM EQUATION FOR STRATIFIED FLOW

The local instantaneous balance of momentum averaged over the cross sectional area occupied by phase k in a pipe and projected along the tube axis (Delhaye 1981), assuming that the square of the average velocity of phase k is equal to the average of the square of

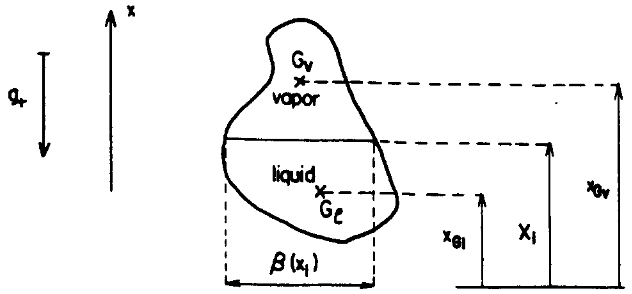


Figure 1. Definition of β (x_l), x_{Gl}, x_l and x_{Gv}

the velocity of phase k, and using H2, is

$$
\begin{aligned}
&\frac{\partial}{\partial t} A\, \alpha_k\, \rho_k\, V_k + \frac{\partial}{\partial z} A\, \alpha_k\, \rho_k\, V_k^2 + \frac{\partial}{\partial z} A\, \alpha_k < p_k > \\
&+ \int_{\mathscr{C} \cup \mathscr{C}_k(z,t)} \mathbf{n}_z \cdot \mathbf{n}_k\, p_k \frac{\mathrm{d}\mathscr{C}}{\mathbf{n}_k \cdot \mathbf{n}_{k\mathscr{C}}} - \frac{\partial}{\partial z} A\, \alpha_k < (\mathbf{n}_z \cdot \mathfrak{S}_k) \cdot \mathbf{n}_z > \\
= {} & A\, \alpha_k\, \rho_k < \mathbf{F}_k \cdot \mathbf{n}_z > + \int_{\mathscr{C}(z,t) \cup \mathscr{C}_k(z,t)} \mathbf{n}_z \cdot (\mathbf{n}_k \cdot \mathfrak{S}_k) \frac{\mathrm{d}\mathscr{C}}{\mathbf{n}_k \cdot \mathbf{n}_{k\mathscr{C}}} \\
&- \int_{\mathscr{C}(z,t)} \mathbf{n}_z \cdot V_k\, \dot{m}_k \frac{\mathrm{d}\mathscr{C}}{\mathbf{n}_k \cdot \mathbf{n}_{k\mathscr{C}}} .
\end{aligned}
\qquad [3.1]
$$

Notations $\mathscr{C}_k$, $\mathscr{C}$, $\mathbf{n}_k$, $\mathbf{n}_{k\mathscr{C}}$, $\mathbf{n}_z$ are explained in figure 2.

A is the cross section area, $\mathfrak{S}_k$ the viscous stress tensor of phase k, $\mathbf{F}_k$ the external force per unit of mass (i.e. gravity or acceleration of entrainment), and $\dot{m}_k$ the mass transfer per unit area of interface and per unit of time.

Neglecting the axial shear stress effect ($(\partial/\partial z)\, A\, \alpha_k < (\mathbf{n}_z \cdot \mathfrak{S}_k) \cdot \mathbf{n}_z >$ is neglected), some calculations, set out in detail in appendix A, lead to the following expression of the balance of momentum equation:

$$
\begin{aligned}
&\frac{\partial}{\partial t} A\, \alpha_k\, \rho_k\, V_k + \frac{\partial}{\partial z} A\, \alpha_k\, \rho_k\, V_k^2 + A\, \alpha_k \frac{\partial P}{\partial z} + A\, p_i \frac{\partial \alpha_k}{\partial z} + A\, \epsilon_k\, E_l \frac{\partial V_l}{\partial z} \\
&+ A\, \epsilon_k\, E_v \frac{\partial V_v}{\partial z} + A\, \epsilon_k\, N \frac{\partial M}{\partial z} + \cdots = -A\, \epsilon_k\, G_z - D_k - \alpha_k\, \Delta_k \frac{\partial A}{\partial z} \\
&+ A\, \alpha_k\, \rho_k < \mathbf{F}_k \cdot \mathbf{n}_z > + A\, \epsilon_k\, \Gamma\, W_i - A\, \epsilon_k\, \tau_i - C_k\, \chi_f\, \tau_k \quad .
\end{aligned}
\qquad [3.2]
$$

Equation [3.2] is also [A.7] of appendix A, and

— Γ is the rate of mass transfer between phases per unit volume of the duct,
— W_i is the velocity of the interface,
— τ_i is the interfacial shear stress per unit volume projected on the tube axis,
— χ_f is the frictional perimeter,
— τ_k is the wall shear stress per unit of surface projected on the tube axis for phase k,
— C_k is the dimensionless coefficient of influence of friction of phase k.

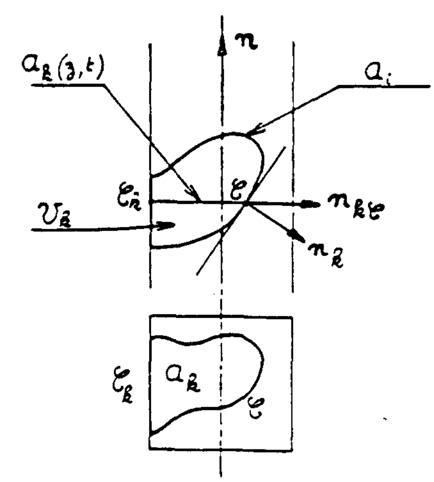


Figure 2. Notations used in [3.1].

We use some specific terms:

$$p_i \triangleq \alpha\, \Delta_l + (1-\alpha)\, \Delta_v + \alpha(1-\alpha) \frac{\partial}{\partial \alpha} (\Delta_v - \Delta_l)\,, \quad [3.3]$$

$$E_v \triangleq \alpha(1-\alpha) \frac{\partial}{\partial V_v} (\Delta_v - \Delta_l)\,, \quad [3.4]$$

$$E_l \triangleq \alpha(1-\alpha) \frac{\partial}{\partial V_l} (\Delta_v - \Delta_l), \quad [3.5]$$

$$N \triangleq \alpha(1-\alpha) \frac{\partial}{\partial M} (\Delta_v - \Delta_l)\,, \quad [3.6]$$

$$G_z \triangleq \alpha(1-\alpha) \frac{\partial}{\partial z} (\Delta_v - \Delta_l)\,, \quad [3.7]$$

$$D_k \triangleq \int_{\mathscr{C}_k(z,t)} \mathbf{n}_z \cdot \mathbf{n}_k\, \delta_k \frac{\mathrm{d}\mathscr{C}}{\mathbf{n}_k \cdot \mathbf{n}_k}\,.$$

p_i represents the effects of the variation of the level, and E_v, E_l, N and G_z represent the axial effect of the variation of the apparent transverse gravity with V_v, V_l, M and z. D_k represents the effect of the axial variation of the cross-section shape and area.

4. ENERGY EQUATION FOR STRATIFIED FLOW

The same method is used for energy equations in appendix B, introducing similar specific terms. As these terms are usually not numerically predominant, their main interest in the energy equation is to point out the resemblance to the momentum equation.

Another important consequence of this model is that the heat transfer between phases (important in a two-phase flow pump where the pressure may change significantly along the axis) must be calculated with phase properties at the pressure P_i at the interface and not mean pressure P. Some special cases of cavitation can be studied with this modelisation (P_i, being usually lower than P, can be lower than saturation pressure for the mean liquid temperature for a pump with single phase at the inlet).

5. REMARKS

Specific terms due to the apparent transverse gravity, which appear in the momentum and energy written with fairly general assumptions, do not all appear together. This will be shown in sections 6 to 9.

Let us note that the term p_i, known in classic earth gravity stratified two-phase flow, is not an interfacial pressure deficiency, as it has been named by Andreoni *et al.* (1981). Indeed,

$$p_i \neq P - P_i$$

These specific terms, except $-D_k - \alpha_k\, \Delta_k\, (\partial A / \partial z)$, represent pressure interactions between the phases. They disappear in mixture equations (sum of the phase equations). They are numerically important mainly in the vapor momentum equation. Physically, these terms are due to pressure interactions and represent the effects of the axial gradient of apparent transverse gravity. Indeed, it may seem surprising that forces acting perpendicularly to the axis of the tube have such an important effect projected on the axis of the tube. This may be understood with the help of some examples:

(a) Classic stratified flow due to earth's gravity

The earth's gravity, perpendicular to the axis of the flow, has an effect projected on the axis of the flow. (Arrows in figure 3a represent the direction of the effect.)

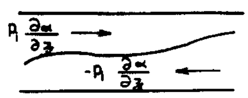


Figure 3a. Out of equilibrium, $p_i\,(\partial\alpha/\partial z) \neq 0$.

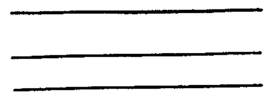


Figure 3b. In equilibrium, $p_i\,(\partial\alpha/\partial z) = 0$.

(b) Stratified flow in variable gravity

Let us imagine a stratified two-phase flow on a planet where gravity, constant in direction, is variable in intensity.

This gravity, variable but perpendicular to the axis of the flow, has an effect projected on the axis of the flow. (Arrows in figure 4a represent the direction of the effect.)

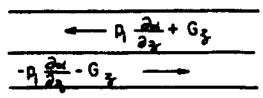


Figure 4a. Out of equilibrium. $G_z \neq 0$ $p_i\,(\partial\alpha/\partial z) = 0$.

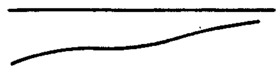


Figure 4b. In equilibrium. $p_i\,(\partial\alpha/\partial z) + G_z = 0$.

(c) Stratified flow in velocity dependent gravity

Let us imagine a stratified two-phase flow in a rotating device, where Coriolis forces are important and where gravity is proportional and perpendicular to the velocity.

This gravity, perpendicular to the axis of the flow, has an important effect projected on the axis of the flow. (Arrows in figures 5a and 5b represent velocity.)

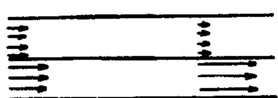


Figure 5a. Out of equilibrium. $E_v\,(\partial V_v/\partial pz) + E_l\,(\partial V_l/\partial z) \neq 0$.

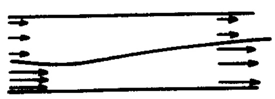


Figure 5b. In equilibrium. $E_v\,(\partial V_v/\partial z) + E_l\,(\partial V_l/\partial z) + p_i\,(\partial\alpha/\partial z) = 0$.

Examples of applications

In order to explain the meaning and the consequences of these specific terms four examples will be presented, but others may be imagined.

6. EXAMPLE 1 — "CLASSIC" STRATIFIED FLOW (PIPE IN THE EARTH'S GRAVITY FIELD)

In this classic case, we have

$$F_{k,x} = g \qquad \text{and} \qquad F_{k,y} = 0 \quad , \tag{6.1}$$

where g is the absolute value of the component of earth's gravity perpendicular to the tube axis.

We have

$$\Delta_k = -\,\rho_k\, g\,(x_{G,k} - X_i) \tag{6.2}$$

and can show that

$$p_i = \alpha(1-\alpha)\,\frac{A}{\beta\,(X_i)}\, g\,(\rho_l - \rho_v) \quad , \tag{6.3}$$

where $\beta(X_i)$ is defined with

$$E_v = E_l = 0 \quad , \quad N = 0 \quad , \quad G_z = 0 \quad . \tag{6.4}$$

If the shape of the duct cross section is rectangular, with the side of dimension D parallel to the component of earth's gravity, we have

$$\alpha_k \Delta_k \frac{\partial A}{\partial z} + D_k = \frac{\epsilon_k}{2} \alpha (1-\alpha) \rho_k g A \frac{\partial D}{\partial z} \qquad [6.5]$$

and

$$G_z = -\frac{1}{2} \alpha(1-\alpha) (\alpha \rho_v + (1-\alpha) \rho_l) \frac{\partial}{\partial z} (g D) \quad . \qquad [6.6]$$

The characteristic equation is found from the characteristic analysis of the mass equation [B.5] and the momentum equation [3.2]. In the present case, assuming that the constitutive equations for Γ, τ_i, τ_k and W_i do not depend on the derivatives of the dependent variables (in particular, we suppose we have no added mass effects), and also assuming that ρ_v and ρ_l are constant, the characteristic equation is

$$\lambda^2 \rho - 2 \lambda \widetilde{\rho V} + \widetilde{\rho V^2} - p_i = 0 \quad . \qquad [6.7]$$

The notation $\tilde{\chi}$ for a phase variable χ_k or a product of phase variables represents the cross mean value

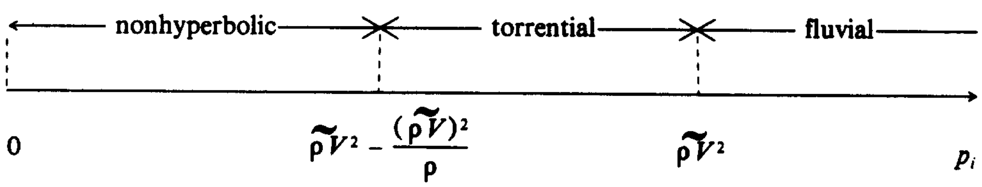
$$\tilde{\chi} = \alpha \chi_l + (1-\alpha) \chi_v \quad .$$

The solution of [6.7] are the eigenvalues. We can define three flow patterns:

Nonhyperbolic flow. The two eigenvalues are complex, and the system is not hyperbolic. That means that small perturbations of infinite wavenumber and infinite frequency are not stable. This corresponds to the classical linear instability. The practical consequence is that the flow cannot remain stratified.

Torrential flow. The two eigenvalues are real and of the same sign. That means that if the flow is stratified, small disturbances cannot move upstream.

Fluvial flow. The two eigenvalues are real and of opposite sign. That means that if the flow is stratified, small disturbances can move upstream.



The hyperbolic condition may be written either

$$p_i > \widetilde{\rho V^2} - \frac{(\widetilde{\rho V})^2}{\rho} \qquad [6.8]$$

or equivalently

$$(V_v - V_l)^2 < \frac{\tilde{\rho}(\rho_l - \rho_v) A g}{\rho_l \rho_v \beta(X_l)} \quad . \qquad [6.9]$$

Several phenomena may be studied with this model, such as wave propagation and counter-current stratified flow, small hydraulic jumps, and stratified flow in an horizontal venturi or nozzle. For example, the difference in velocity ($V_v - V_l$), deduced from [6.9], differs by a factor 2 with respect to the experimental limit of stratification of Wallis (1973) or the theoretical limit of Mishima & Ishii (1980).

7. EXAMPLE 2 — CURVED PIPE

In this example we deal with a regularly curved pipe, such as a helix tube or the casing of a pump, with stratified flow caused by centrifugal forces. This study does not apply to bends in which entry effects are predominant.

For the purpose of simplification, the earth's gravity is neglected. The apparent transverse gravity is the centifugal force.

We suppose that the x-axis is in the osculating plane of the tube axis, and the apparent transverse gravity is

$$F_{k,x} = - V_k^2 C, \quad F_{k,y} = 0 \quad , \qquad [7.1]$$

where C is the curvature of the pipe, positive if the curvature is directed toward the positive direction of the x-axis.

We have

$$\Delta_k = -\rho_k V_k^2 C (x_{G,k} - X_i) \qquad [7.2]$$

and

$$p_i = -\alpha(1-\alpha) \frac{A}{\beta(X_i)} C (\rho_l V_l^2 - \rho_v V_v^2) \quad , \qquad [7.3]$$

where $\beta(X_i)$ is defined as for the classical stratified flow, figure 1.

We have

$$E_k = \alpha(1-\alpha)\, \rho_k K_k \quad , \qquad [7.4]$$

where K_k is defined by

$$K_k \triangleq + 2\, \epsilon_k (x_{G,k} - X_i)\, V_k C \quad . \qquad [7.5]$$

In the particular case of a rectangular pipe, with one of the sides of dimension D parallel to the x-axis, we have

$$K_k = + \alpha_k DC V_k \quad . \qquad [7.6]$$

The characteristic equation is found from the characteristic analysis of mass equations [B.5] and momentum equation [3.2]. In the present case, we assume that the constitutive equations for Γ, τ_i, τ_k and W_i do not depend on the derivatives of the dependent variables (in particular, we suppose we have no added mass effects).

If ρ_v and ρ_l are constant, the characteristic equation is

$$\lambda^2 \tilde{\rho} - \lambda (2 \widetilde{\rho V} + \widetilde{\epsilon \rho K}) + \widetilde{\rho V^2} + \widetilde{\epsilon \rho K V} - p_i = 0 \quad , \qquad [7.7]$$

where the notation $\tilde{}$ is explained in section 6.

In the particular case of a rectangular pipe, we have

$$p_i = \widetilde{\epsilon \rho K V} \quad , \qquad [7.8]$$

and the flow is always torrential or nonhyperbolic. An eigenvalue analysis shows that as the quantity

$$\left| \frac{\rho_l V_l - \rho_v V_v}{\rho_v V_v} \right|$$

grows, the eigenvalues become real.

8. EXAMPLE 3 — PRE- OR POSTROTATION IN THE IMPELLER OF A PUMP OR A TURBINE

When a pump (or turbine) operates outside its nominal delivery, a swirling reverse flow appears in the inlet duct (cf. figure 6). Below the nominal delivery, this swirling reverse flow rotates in the same direction as impeller rotation, and in the opposite direction above nominal delivery. This phenomenon is called "prerotation" for positive flow or "postrotation" for negative flow. This pre- or postrotation causes centrifugal forces, equivalent to an apparent transverse gravity, which bring about an annular flow: liquid near the wall, vapor in the center of the duct.

In order to give analytical expressions of the supplementary terms, we assume that the cross section of the suction duct is cylindrically symmetric, and we have to choose a radial distribution of tangential velocity. In order to simplify calculations it is assumed that the tangential velocity is proportional to the radius. This means that each point rotates at the same angular velocity (solid rotation).

Noting that $V_{k,T}$ is the tangential velocity of phase k, the radial pressure gradient is given by

$$\frac{\partial p_k}{\partial r} = \rho_k \frac{V_{k,T}^2}{r} \quad , \qquad [8.1]$$

where radius r and angle θ are the polar coordinates and [8.1] is equivalent to [2.1], but in polar coordinates, with

$$F_{k,r} = \frac{V_{k,T}^2}{r} \quad \text{and} \quad F_{k,\theta} = 0 \quad . \qquad [8.2]$$

We introduce the total angular momentum M, defined by

$$M = \int_{A_v} \rho_v\, r\, V_{v,T}\, \mathrm{d}A + \int_{A_l} \rho_l\, r\, V_{l,T}\, \mathrm{d}A \quad .$$

M may be regarded as a 7th main variable (the overall balance of angular momentum being the 7th equation).

With the notations

$$\rho \triangleq \alpha^2 \rho_v + (1-\alpha)^2 \rho_l \quad \text{and} \quad \bar{\rho} = \alpha\, \rho_v + (1-\alpha)\, \rho_l \qquad [8.3]$$

we have

$$\Delta_k = -\frac{\pi M^2}{A^3} \epsilon_k \alpha_k \frac{\rho_k}{\bar{\rho}^2} \quad , \qquad [8.4]$$

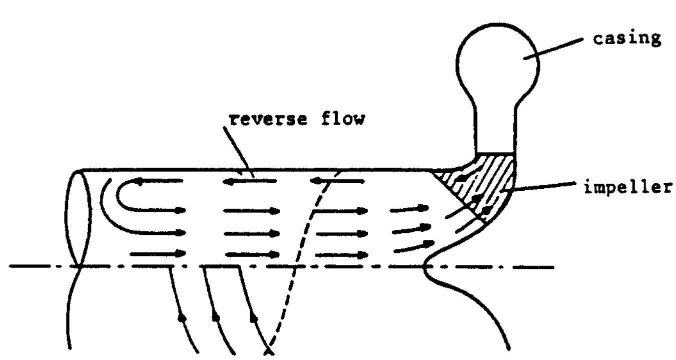


Figure 6. Prerotation and reverse flow. Schematisation of the flow in the inlet duct at low delivery.

$$p_i = \frac{2\pi M^2}{A^3}\,\alpha(1-\alpha)\,(\rho_l - \rho_v)\,\frac{|(1-\alpha)^2\,\rho_l - \alpha^2\,\rho_v|}{\hat{\rho}^3} \quad , \qquad [8.5]$$

$$N = -\,\alpha(1-\alpha)\,\frac{2\,\pi\,M}{A^3}\,\frac{\bar{\rho}}{\hat{\rho}^2} \quad , \qquad [8.6]$$

$$G_z = \alpha(1-\alpha)\,\frac{3\,\pi\,M^2}{A^4}\,\frac{\bar{\rho}}{\hat{\rho}^2}\cdot\frac{\mathrm{d}A}{\mathrm{d}_z} \quad , \qquad [8.7]$$

$$G_l = 0,\; E_v = E_l = 0 \quad , \qquad [8.8]$$

$$D_v = 0,\; D_l = -\,2\,\Delta_l\,\frac{\mathrm{d}A}{\mathrm{d}z} \quad . \qquad [8.9]$$

The supplementary terms are numerically predominant in the vapor momentum equation.

An eigenvalue analysis shows that eigenvalues are real and stability is maximum for low void fractions (de Crecy 1983). This modelisation shows that the annular stratified flow due to pre- or postrotation is always unstable for high void fractions. This is consistant with the experimental observations: The author does not know of any experimental evidence of high void fractions in annular stratified flow due to pre- or postrotation, but numerous observations for low void fractions have been made. Overpressure on the wall, with respect to the mean pressure, may be several bars in postrotation. It is very important to take this fact into account when the performances of pumps with two-phase flow are measured.

Taking into account pre- or postrotation in the suction duct of a pump leads to a slip ratio and a void fraction at the inlet of the impeller very different from those obtained with the classical modelisation without effects of the pre- or postrotation. The inlet slip ratio and the inlet void fraction have an important effect on the stratification in the impeller and, consequently, on the head of the two-phase flow pump.

Another application of this example could be the annular cavitation in the discharge pipe of a turbine caused by postrotation. This phenomenon could be studied using classical correlations for heat and mass transfer with the property of water and steam at the pressure P_i and not P.

9. EXAMPLE 4—FLOW IN THE IMPELLER OF A PUMP

It has been seen in the introduction that the experimenters conclude that the head degradation of a pump seems to be caused by stratification in the impeller. This can be easily explained. Let us consider a two-phase flow in an inclined pipe in a gravity field (figures 7a and b).

In the nonstratified inclined flow (figure 7a), the phases are strongly coupled and velocities in cross section A and B are quite similar. There is conversion of the gravitational potential energy of the flow in cross section A to the pressure in cross section B.

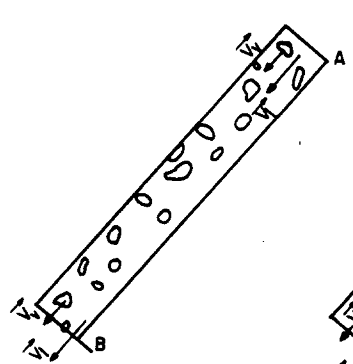


Figure 7a. Nonstratified inclined flow in a gravity field.

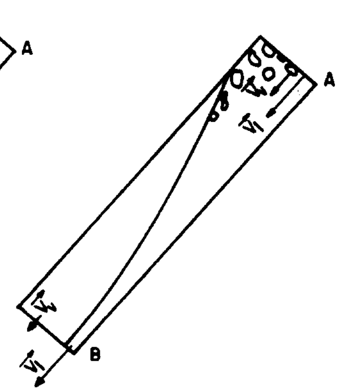


Figure 7b. Stratified inclined flow in a gravity field.

On the other hand, in the stratified inclined flow (figure 7b) the phases are more independent and velocities in cross section A are very different from those in cross section B. There is conversion of the potential energy of the flow in cross section A to the kinetic energy of the liquid in cross section B.

The same phenomena lead to a conversion of the energy transferred by the motor of the pump to pressure at the outlet of the impeller if the flow is not stratified (consequently, there is no important head degradation) or to kinetic energy in the liquid at the outlet of the impeller if the flow is stratified. In this last case, the diffuser and the casing are usually maladapted and only a very small transformation of kinetic energy to piezometric energy occurs in it; there is an important head degradation of the pump.

In order to understand and predict this phenomenon, it is important to study:

(a) The limit of stratification, which is not the purpose of this paper.

(b) The stratified flow in the impeller of the pump.

The study of the stratified flow in the impeller may be used to explain the effects of the geometry of the impeller (consequently the effect of the specific speed), of the inlet void fraction or inlet slip ratio, and so on.

Four stratifying forces act as apparent transverse gravity in an axial description of two-phase flow in a pump. Ω is the angular velocity of the impeller, and X represent the vector product (cross product).

(a) Perpendicular component of centrifugal forces and acceleration of entrainment: $-\,\Omega\,X\,(\Omega\,X\,\mathbf{R}) - (\mathrm{d}\Omega/\mathrm{d}t)\,X\,\mathbf{R}$.

(b) Coriolis forces: $-\,2\,\Omega\,X\,\mathbf{V}_k$.

(c) Curvature effects: $-\,V_k^2\,(\mathrm{d}\mathbf{n}_z/\mathrm{d}z)$.

(d) Earth gravity: $\mathbf{g}$.

Usually, centrifugal and Coriolis forces are predominant and earth's gravity is negligible, and the apparent transverse gravity may reach thousands of times that of the earth.

Centrifugal forces and Coriolis forces act in the opposite direction if $\omega\,Q > 0$, in the same direction if $\omega\,Q < 0$. (Q being the total volumetric flow rate and the speed of rotation).

The impeller may be described in cylindrical coordinates, where $(\mathbf{n}_r, \mathbf{n}_\theta, \mathbf{n}_\omega)$ is the local base of orthonormal vectors ($\mathbf{n}_r$ is radial, $\mathbf{n}_\theta$ is tangential and $\mathbf{n}_\omega$ is colinear to Ω). The flow will be described in the local non-Gallilean frame of reference, using the center of inertia of the cross section as the origin and an orthonormal base of vectors, $(\mathbf{n}_x, \mathbf{n}_y, \mathbf{n}_z)$, defined by:

$$\begin{aligned}
&\mathbf{n}_z \text{ is in the direction of the mean velocity in nominal conditions} \quad ,\\
&\mathbf{n}_x \text{ is normal to } \mathbf{n}_\omega \quad ,\\
&\mathbf{n}_y = \mathbf{n}_z\,X\,\mathbf{n}_x \quad ,\\
&\mathbf{n}_x \text{ is chosen in order to have } \mathbf{n}_y \cdot \mathbf{n}_\omega > 0 \quad .
\end{aligned} \qquad [9.1]$$

The angles γ and ψ are defined by

$$\begin{aligned}
\gamma &\triangleq (\mathbf{n}_\omega, \mathbf{n}_y) \quad ,\\
\psi &\triangleq (\mathbf{n}_x, \mathbf{n}_\theta) \quad .
\end{aligned} \qquad [9.2]$$

Let R be the mean radius (i.e. the distance between the axis of rotation and the center of inertia of the cross section) and g the projection on $\mathbf{n}_\omega$ of the earth's gravity.

With these notations, the following apparent gravity is obtained:

$$\frac{1}{\rho_k}\frac{\partial p_k}{\partial x} = -\,R\,\frac{\mathrm{d}\omega}{\mathrm{d}t}\cos\psi + \omega^2 R \sin\psi - 2\,\omega\,V_k\cos\gamma + V_k^2\cos\gamma\left(\frac{\mathrm{d}\psi}{\mathrm{d}z} + \frac{1}{R}\sin\psi\cos\gamma\right) + x\,\omega^2 \qquad [9.3]$$

$$\frac{1}{\rho_k}\frac{\partial p_k}{\partial y} = g\cos\gamma - R\,\frac{\mathrm{d}\omega}{\mathrm{d}t}\sin\psi\sin\gamma - \omega^2 R\cos\psi\sin\gamma - V_k^2\left(\frac{\mathrm{d}\gamma}{\mathrm{d}z}\right) + y\,\omega^2\sin^2\gamma. \qquad [9.4]$$

Let $F'_{k,x}$ and $F'_{k,y}$ be the part of the apparent transverse gravity independent of x and y:

$$F'_{k,x} \triangleq -R\frac{d\omega}{dt}\cos\gamma + \omega^2 R\sin\psi - 2\,\omega\,V_k\cos\gamma + V_k^2\cos\gamma\left(\frac{d\psi}{dz} + \frac{1}{R}\sin\psi\cos\gamma\right), \quad [9.5]$$

$$F'_{k,y} \triangleq g\cos\gamma - R\frac{d\omega}{dr}\sin\psi\sin\gamma - \omega^2 R\cos\psi\sin\gamma - V_k^2\left(\frac{d\gamma}{dz}\right). \quad [9.6]$$

According to [2.3], [2.4], [9.3] and [9.4], we have

$$\Delta_k \triangleq \rho_k\left[F'_{k,x}(x_{G,k}) + F'_{k,y}(y_{G,k} - Y_i) + \frac{1}{2}\omega^2(x^2_{m,k} - X_i^2) + \frac{1}{2}\omega^2\sin^2\gamma\,(y^2_{m,k} - Y_i^2)\right], \quad [9.7]$$

where $(x_{m,k}, y_{m,k})$ is the center of moment of inertia of phase k with respect to the center of inertia of the cross section.

If the shape of the cross section is known it is possible to determine analytically the expression of Δ_v and Δ_l as a function of α, V_v, V_l, ρ_v, ρ_l, ω and of the geometrical data and also an analytical expressions of E_v, E_l, p_i, G_z, G_t, D_v, D_l. In the general case, these expressions are very complicated. The terms E_v and E_l contain effects of the axial variation of Coriolis forces and curvature effects.

The main part of G_z is caused by the axial variation of the perpendicular component of centrifugal forces, and the main part of G_t is proportional to $d\omega/dt$, *i.e.* to the rate of change of apparent transverse gravity.

In the applications computed by the author, the term p_i is usually the numerically predominant term. It contains effects of the three major apparent transverse gravities (centrifugal forces, Coriolis forces, curvature effects) and their axial variations.

10. COMPARISON WITH A VERY SIMPLE EXPERIMENT

Rundstadler *et al.* (1978) have conducted an experiment with a single constant cross-sectional area, square-shaped, rotating Plexiglas channel with air and water flow (figure 8).

As our model is not yet available on computer, we have chosen this experiment as one of the simplest possible. Theoretical results of the present model show good agreement with

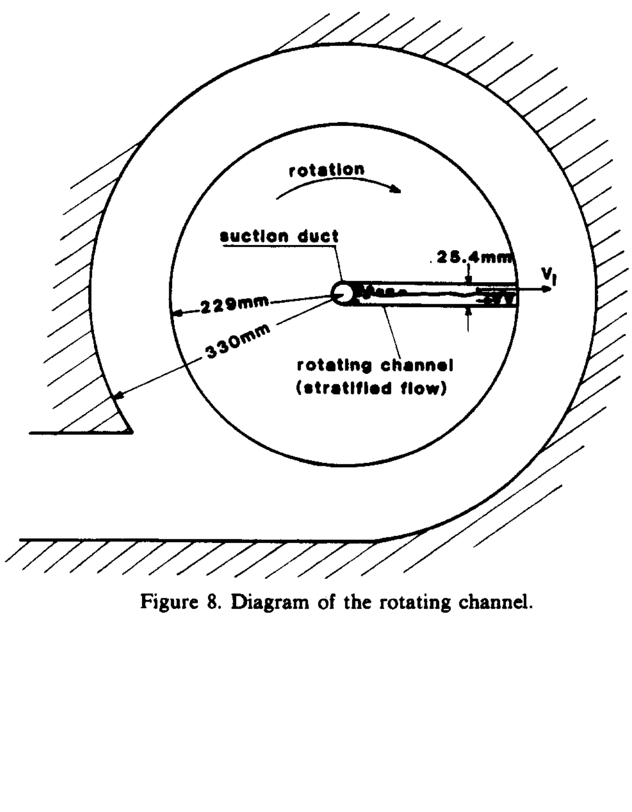


Figure 8. Diagram of the rotating channel.

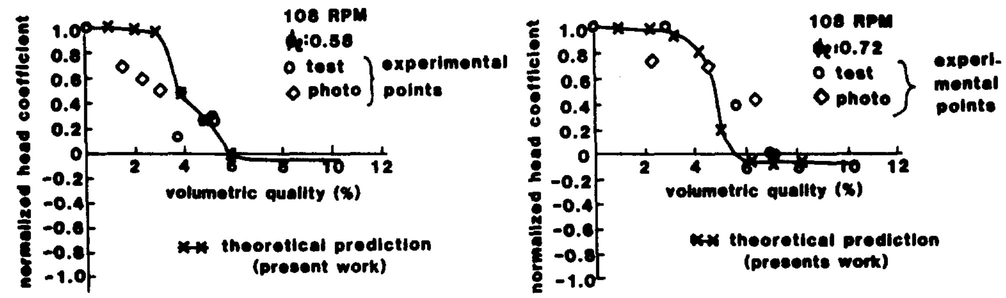


Figure 9. Two-phase performance of the rotating channel.

experimental data (figure 9). The Y-axis represents the normalised head coefficient, defined by $H_{2\phi}/H_o$ where $H_{2\phi}$ is the actual two-phase head and H_o the single-phase head for the same total volumetric flow rate. The X-axis represents the volumetric quality $j_v/(j_v + j_l)$, where j_v and j_l are the superficial velocity of vapor and liquid, and ϕ_l is a dimensionless volumetric flow rate of liquid, defined by

$$\phi_l \triangleq \frac{j_l}{\omega R} \quad ,$$

where R is the external (outlet) radius and ω the speed of rotation.

Rundstadler & Patel (1978) made photos of the rotating channel when the head is degraded. These photos show that the flow stratified very quickly and that the void fracations seems to be .4 or .5 in the middle of the channel for a volumetric quality of .03 to .07. The volumetric quality for which the head is fully degraded (i.e. where $P=0$ in the rotating channel) is quite sensitive to the limit of stratification used in the calculations made with the present model.

11. CONCLUSIONS

In this paper, we proposed a description of a stratified flow in a general case by a one-dimensional, six-equation model. This one-dimensional model includes some three-dimensional information, such as the effects of the apparent transverse gravity and its variations. The effects of pressure interactions between phases are explicitly taken into account by specific terms in momentum and energy equations. These terms may be numerically predominant, especially in the vapor momentum equation.

Four examples were discussed: classic earth's gravity stratified flow, stratified flow in a regularly curved pipe, annular stratified flow caused by pre- or postrotation of a pump, and stratified flow in the impeller of a pump. Some comparisons with a very simple experiment show good agreement with this theory.

Future work should include comparisons with numerous experimental data. Additional experimental and theoretical work is also needed to establish a valid and reliable criterion of stratification for examples 2, 3, and 4.

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APPENDIX A

Momentum equation for stratified flow: Detailed calculations

Equation [3.1] is the starting point for these calculations. The first objective is to use only one mean pressure P and not two-phase pressures $<p_k>$, and to develop the term

$$\int_{\mathscr{C}U\mathscr{C}_k(z,t)} \mathbf{n}_z \cdot \mathbf{n}_k p_k \frac{\mathrm{d}\mathscr{C}}{\mathbf{n}_k \cdot \mathbf{n}_{k\mathscr{C}}} \quad .$$

As the interfacial pressure P_i is constant over the interface (assumption H5), we have

$$\int_{\mathscr{C}U\mathscr{C}_k(z,t)} \mathbf{n}_z \cdot \mathbf{n}_k p_k \frac{\mathrm{d}\mathscr{C}}{\mathbf{n}_k \cdot \mathbf{n}_{k\mathscr{C}}} = P_i \int_{\mathscr{C}U\mathscr{C}_k(z,t)} \mathbf{n}_z \cdot \mathbf{n}_k \frac{\mathrm{d}\mathscr{C}}{\mathbf{n}_k \cdot \mathbf{n}_{k\mathscr{C}}} + \int_{\mathscr{C}_k(z,t)} \mathbf{n}_z \cdot \mathbf{n}_k \delta_k \frac{\mathrm{d}\mathscr{C}}{\mathbf{n}_k \cdot \mathbf{n}_{k\mathscr{C}}} \quad [A.1]$$

with notations explained in sections 2 and 3. As a consequence of the limiting form of the Leibniz rule (Delhaye 1981), the first part of the right hand term is equal to $-P_i\,(\partial/\partial z)\,(A\alpha_k)$.

Let us define

$$D_k \triangleq \int_{\mathscr{C}_k(z,t)} \mathbf{n}_z \cdot \mathbf{n}_k\, \delta_k \frac{\mathrm{d}\mathscr{C}}{\mathbf{n}_k \cdot \mathbf{n}_k \mathscr{C}} \quad . \quad [A.2]$$

We then have

$$\int_{\mathscr{C}U\mathscr{C}_k(z,t)} \mathbf{n}_z \cdot \mathbf{n}_k p_k \frac{\mathrm{d}\mathscr{C}}{\mathbf{n}_k \cdot \mathbf{n}_{k\mathscr{C}}} = -P_i \frac{\partial}{\partial z}(A\,\alpha_k) + D_k \quad . \quad [A.3]$$

In the particular case where area and shape of a cross section are constant, we have

$$\mathbf{n}_z \cdot \mathbf{n}_k = 0 \text{ on } \mathscr{C}_k$$

and consequently

$$D_k = 0 \quad ,$$

with the notation explained in section 3.

$$-\int_{\mathscr{C}(z,t)} \mathbf{m}_z \cdot \mathbf{V}_k \dot{m}_k \frac{\mathrm{d}\mathscr{C}}{\mathbf{n}_k \cdot \mathbf{n}_{k\mathscr{C}}} = \epsilon_k A\, \Gamma\, W_i \quad [A.4]$$

is the momentum transfer due to mass transfer.

$$+\int_{\mathscr{C}(z,t)} \mathbf{n}_z \cdot (\mathbf{n}_k \cdot \mathfrak{S}_k) \frac{\mathrm{d}\mathscr{C}}{\mathbf{n}_k \cdot \mathbf{n}_{k\mathscr{C}}} = -\epsilon_k A\, \tau_i \quad [A.5]$$

is the interfacial shear stress.

$$\int_{\mathscr{C}_k(z,t)} \mathbf{n}_z \cdot (\mathbf{n}_k \cdot \mathfrak{S}_k) \frac{\mathrm{d}\mathscr{C}}{\mathbf{n}_k \cdot \mathbf{n}_{k\mathscr{C}}} = -C_k\, \chi_f\, \tau_k \quad [A.6]$$

is the wall shear stress.

Equation [3.1] becomes

$$\frac{\partial}{\partial t} A\,\alpha_k\,\rho_k\,V_k + \frac{\partial}{\partial z} A\,\alpha_k\,\rho_k\,V_k^2 + \frac{\partial}{\partial z} A\,\alpha_k <p_k> - P_i \frac{\partial}{\partial z}(A\,\alpha_k)$$
$$= -D_k + A\,\alpha_k\,\rho_k <\mathbf{F}_k\cdot\mathbf{n}_z> + \epsilon_k A\,\Gamma\,W_i - C_k\,\chi_f\,\tau_k \quad . \qquad [A.7]$$

We also have

$$\frac{\partial}{\partial z}(A\,\alpha_k <p_k>) - P_i \frac{\partial}{\partial z}(A\,\alpha_k) + D_k$$
$$= A\left[\frac{\partial}{\partial z}\alpha_k <p_k> - P_i \frac{\partial \alpha_k}{\partial z}\right] + \alpha_k\,\Delta_k \frac{\partial A}{\partial z} + D_k \quad . \qquad [A.8]$$

As Δ_k may be a function of α, V_v, V_l, M, z, or any other variable, we have, using [2.5] and [2.9],

$$\begin{aligned}
\frac{\partial}{\partial z}\alpha_k <p_k> - P_i \frac{\partial \alpha_k}{\partial z} &= \alpha_k \frac{\partial P}{\partial z} \\
&+ \left[\alpha\,\Delta_l + (1-\alpha)\,\Delta_v + \alpha(1-\alpha)\frac{\partial}{\partial \alpha}(\Delta_v - \Delta_l)\right]\frac{\partial \alpha_k}{\partial z} \\
&+ \epsilon_k\,\alpha(1-\alpha)\frac{\partial}{\partial V_l}(\Delta_v - \Delta_l)\cdot\frac{\partial V_l}{\partial z} \\
&+ \epsilon_k\,\alpha(1-\alpha)\frac{\partial}{\partial V_v}(\Delta_v - \Delta_l)\cdot\frac{\partial V_v}{\partial z} \\
&+ \epsilon_k\,\alpha(1-\alpha)\frac{\partial}{\partial M}(\Delta_v - \Delta_l)\cdot\frac{\partial M}{\partial z} \\
&+ \epsilon_k\,\alpha(1-\alpha)\frac{\partial}{\partial z}(\Delta_v - \Delta_l) \\
&+ \cdots .
\end{aligned} \qquad [A.9]$$

Using terms such as p_i, E_v, E, N, G_z defined from [3.3] to [3.6], [A.7] is transformed by [A.9] into [3.2].

APPENDIX B

Energy equation for stratified flow: Detailed calculations

In order to simplify the calculations, we make a supplementary assumption:

H6: the walls are impermeable and rigid, which means

$$\mathbf{V}_k\cdot\mathbf{n}_k = 0 \text{ on } \mathscr{C}_k(z,t). \qquad [B.1]$$

The local instantaneous energy balance averaged over the cross-sectional area occupied by phase k is (Delhaye 1981)

$$\begin{aligned}
&\frac{\partial}{\partial t} A\,\alpha_k \left\langle \rho_k\left(\frac{1}{2}V_k^2 + u_k\right)\right\rangle + \frac{\partial}{\partial z} A\,\alpha_k \left\langle \rho_k\,V_k\left(\frac{1}{2}V_k^2 + h_k\right)\right\rangle \\
&- A\,\alpha_k\,\rho_k <\mathbf{F}_k\cdot\mathbf{V}_k> - \frac{\partial}{\partial z} A\,\alpha_k <(\mathfrak{S}_k\cdot V_k)\cdot\mathbf{n}_z> \\
&- \frac{\partial}{\partial z} A\,\alpha_k <\mathbf{q}_k\cdot\mathbf{n}_z> + \int_{\mathscr{C}\cup\mathscr{C}_k(z,t)} p_k\,\mathbf{V}_k\cdot\mathbf{n}_k \frac{\mathrm{d}\mathscr{C}}{\mathbf{n}_k\cdot\mathbf{n}_{k\mathscr{C}}} \\
&- \int_{\mathscr{C}(z,t)}\left[\dot{m}_k\left(\frac{1}{2}V_k^2 + u_k\right) - (\mathfrak{S}_k\cdot\mathbf{V}_k)\cdot\mathbf{n}_k\right]\frac{\mathrm{d}\mathscr{C}}{\mathbf{n}_k\cdot\mathbf{n}_{k\mathscr{C}}} \\
&- \int_{\mathscr{C}_k(z,t)\cup\mathscr{C}(z,t)} \mathbf{q}_k\cdot\mathbf{n}_k \frac{\mathrm{d}\mathscr{C}}{\mathbf{n}_k\cdot\mathbf{n}_{k\mathscr{C}}} \quad ,
\end{aligned} \qquad [B.2]$$

where

u_k is the internal energy of phase k per unit of mass,

h_k is the enthalpy of phase k per unit of mass,

$\mathbf{q}_k$ is the heat flux.

We will neglect the axial shear stress effects (term $(\partial/\partial z)\, A\, \alpha_k <(\ \ _k \cdot \mathbf{V}_k) \cdot \mathbf{n}_z>$) and the variation of axial heat conduction (term $(\partial/\partial z)\, A\, \alpha_k <\mathbf{q}_k \cdot \mathbf{n}_z>$). These terms are usually numerically small compared with the others and would lead to second-order partial differential terms if they were taken into account. In order to simplify equations, we will also assume that covariance coefficient for phase velocities are 1. We assume that

$$<\rho_k\left(\frac{1}{2}\, V_k^2 + u_k\right)> \; = \rho_k\left(\frac{1}{2} <V_k>^2 + <u_k>\right) \tag{B.3}$$

and

$$<\rho_k\, V_k\left(\frac{1}{2}\, V_k^2 + h_k\right)> \; = \rho_k <V_k> \left(\frac{1}{2}<V_k>^2 + <h_k>\right) \quad . \tag{B.4}$$

To simplify notation, $<V_k>$ will be written as V_k, and $<h_k>$ will be written as h_k.

The first objective is to use only one mean pressure P and not two-phase pressures $<p_k>$ and to develop the term

$$\int_{\mathscr{C}U\mathscr{C}_k(z,t)} p_k\, \mathbf{V}_k \cdot \mathbf{n}_k \frac{\mathrm{d}\mathscr{C}}{\mathbf{n}_k \cdot \mathbf{n}_{k\mathscr{C}}} \quad .$$

From assumptions H5 and H6, we have

$$p_k\, \mathbf{V}_k \cdot \mathbf{n}_k = P_i\, \mathbf{V}_k \cdot \mathbf{n}_k \text{ over } \mathscr{C}U\mathscr{C}_k(z,t) \quad . \tag{B.5}$$

Therefore

$$\int_{\mathscr{C}U\mathscr{C}_k(z,t)} p_k\, \mathbf{V}_k \cdot \mathbf{n}_k \frac{\mathrm{d}\mathscr{C}}{\mathbf{n}_k \cdot \mathbf{n}_{k\mathscr{C}}} = P_i \int_{\mathscr{C}U\mathscr{C}_k(z,t)} \mathbf{V}_k \cdot \mathbf{n}_k \frac{\mathrm{d}\mathscr{C}}{\mathbf{n}_k \cdot \mathbf{n}_{k\mathscr{C}}} \quad . \tag{B.6}$$

From the limiting form of the Gauss theorem (Delhaye 1981), we have

$$\int_{\mathscr{C}U\mathscr{C}_k(z,t)} \mathbf{V}_k \cdot \mathbf{n}_k \frac{\mathrm{d}\mathscr{C}}{\mathbf{n}_k \cdot \mathbf{n}_{k\mathscr{C}}} = -\frac{\partial}{\partial z}\int_{A_k(z,t)} \mathbf{n}_z \cdot \mathbf{V}_k \cdot \mathrm{d}A + \int_{A_k(z,t)} \nabla \cdot \mathbf{V}_k\, \mathrm{d}A \quad . \tag{B.7}$$

With assumption H2, the local mass equation becomes

$$\nabla \cdot V_k = 0 \quad . \tag{B.8}$$

From the definition of V_k, we also have

$$\int_{A_k} \mathbf{n}_z \cdot \mathbf{V}_k\, \mathrm{d}A = A\, \alpha_k\, V_k \tag{B.9}$$

and the averaged mass equation with assumption H2 gives (Delhaye 1981)

$$-\frac{\partial}{\partial z} A\, \alpha_k\, V_k = \frac{\partial}{\partial t}(A\, \alpha_k) - \epsilon_k \frac{A\Gamma}{\rho_k} \quad . \tag{B.10}$$

Equation [B.7] becomes

$$\int_{\mathscr{C}U\mathscr{C}_k(z,t)} \mathbf{V}_k \cdot \mathbf{n}_k \frac{\mathrm{d}\mathscr{C}}{\mathbf{n}_k \cdot \mathbf{n}_{k\mathscr{C}}} - \frac{\partial}{\partial t}(A\, \alpha_k) - \epsilon_k \frac{A\Gamma}{\rho_k} \quad , \tag{B.11}$$

and the energy equation, using enthalpy h_k instead of internal energy u_k, becomes

$$\frac{\partial}{\partial t}A\,\alpha_k\,\rho_k\left(\frac{1}{2}V_k^2+h_k\right)+\frac{\partial}{\partial z}A\,\alpha_k\,\rho_k\,V_k\left(\frac{1}{2}V_k^2+h_k\right)$$
$$-\frac{\partial}{\partial t}A\,\alpha_k<p_k>+P_i\frac{\partial}{\partial t}(A\alpha_k) \qquad \text{[B.12]}$$
$$=A\,\alpha_k\,\rho_k\,F_{k,z}\cdot V_k-\chi_c\,q_{p,k}+A\left[q_{k,E}+\epsilon_k\,\Gamma\,h_{k,i}+\epsilon_k\,\Gamma\frac{W_i^2}{2}-\epsilon_k\,\tau_i\,W_i\right]$$

with the notation

$F_{k,z}$ is the projection on the tube axis of the forces per unit of mass acting on phase k,
χ_c is the heated perimeter,
$q_{p,k}$ is the wall heat flux transferred to phase k,
$q_{k,E}$ is the rate of heat transferred per unit of volume from the phase k toward the interface,
$h_{k,i}$ is the enthalpy per unit of mass of phase k at the interface, other quantities have been previously defined.

We are going to carry out the same transformation as in the momentum equations, using in addition the fact that $\partial A/\partial t=0$ (from assumption H6).

As Δ_k may be a function of α, t, V_v, V_l, M and any other variable, we have

$$-\frac{\partial}{\partial t}A\,\alpha_k<p_k>+P_i\frac{\partial}{\partial t}A\,\alpha_k=-A\left\{\frac{\partial}{\partial t}\alpha_k<p_k>-P_i\frac{\partial\alpha_k}{\partial t}\right\} \qquad \text{[B.13]}$$

and

$$\begin{aligned}
&-\frac{\partial}{\partial t}A\,\alpha_k\langle p_k\rangle+P_i\frac{\partial}{\partial t}A\,\alpha_k=-A\,\alpha_k\frac{\partial P}{\partial t}\\
&-A\left\{\left[\alpha\,\Delta_l+(1-\alpha)\,\Delta_v+\alpha(1-\alpha)\frac{\partial}{\partial\alpha}(\Delta_v-\Delta_l)\right]\frac{\partial\alpha_k}{\partial t}\right.\\
&+\epsilon_k\left[\alpha(1-\alpha)\frac{\partial}{\partial V_l}(\Delta_v-\Delta_l)\right]\frac{\partial V_l}{\partial t}\\
&+\epsilon_k\left[\alpha(1-\alpha)\frac{\partial}{\partial V_v}(\Delta_v-\Delta_l)\right]\frac{\partial V_v}{\partial t}\\
&+\epsilon_k\left[\alpha(1-\alpha)\frac{\partial}{\partial M}(\Delta_v-\Delta_l)\right]\frac{\partial M}{\partial t}\\
&\left.+\epsilon_k\left[\alpha(1-\alpha)\frac{\partial}{\partial t}(\Delta_v-\Delta_l)\right]+\cdots\right\}\ .
\end{aligned} \qquad \text{[B.14]}$$

We recognise that the same terms p_i, E_v, E_l, N appear in the momentum equations, where G_z is used instead of G_t.

$$G_t \text{ is defined by: } G_t\stackrel{\Delta}{=}\alpha(1-\alpha)\frac{\partial}{\partial t}(\Delta_v-\Delta_l). \qquad \text{[B.15]}$$

Using terms p_i, E_v, E_l, N, G_t, [B.12] is transformed by [B.14] into

$$\begin{aligned}
&\frac{\partial}{\partial t}A\,\alpha_k\,\rho_k\left(\frac{1}{2}V_k^2+h_k\right)+\frac{\partial}{\partial z}\left(A\,\alpha_k\,\rho_k\,V_k\left(\frac{1}{2}V_k^2+h_k\right)\right)-A\,\alpha_k\frac{\partial P}{\partial t}\\
&-A\,p_i\frac{\partial\alpha_k}{\partial t}-A\,\epsilon_k\,E_v\frac{\partial V_v}{\partial t}-A\,\epsilon_k\,E_l\frac{\partial V_l}{\partial t}-A\,\epsilon_k\,N\frac{\partial M}{\partial t}\\
&=A\,\epsilon_k\,G_t+A\,\alpha_k\,\rho_k\,F_{k,z}\cdot V_k+\chi_c\,q_{p,k}\\
&+A\left[q_{k,E}+\epsilon_k\Gamma\,h_{k,i}+\epsilon_k\Gamma\frac{W_i^2}{2}-\epsilon_k\,\tau_i\,w_i\right]\ .
\end{aligned} \qquad \text{[B.16]}$$